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Lie symmetry classification analysis for nonlinear coupled diffusion

V A Baikov†§, A V Gladkov†‖ and R J Wiltshire‡¶

† Department of Mathematics, State Aviation, Technical University of Ufa, Karl Marks 12, 450000 Ufa, Russia

‡ Division of Mathematics and Computing, University of Glamorgan, Pontypridd CF37 1DL, UK

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Abstract. The group properties of the $(1+1)$-dimensional matrix diffusion equation of the form $\partial y/\partial t = \partial\{\Lambda(y)\partial y/\partial x\}/\partial x$ with respect to point symmetries are given. It is shown that in particular cases the group properties of this equation are similar to those of the corresponding scalar equation. Namely, the Lie algebra is extended for power and exponential functions, with additional extensions when powers of $-\frac{4}{3}$ and -2 exist. However, a specific form of Λ is shown to exist for cases admitting both infinite and finite groups. The result obtained is used to construct new group-invariant solutions for impulsive boundary inputs. Using two similarity variables, a reduction is applied to the coupled diffusion of temperature and volumetric moisture content in porous media under periodic boundary conditions. A perturbation method is employed to obtain an explicit solution for the case when Λ can be expressed exponentially in terms of the similarity variables.

1. Introduction

Whilst systems of pure diffusion equations, in both their linear and nonlinear forms are well known and have many physical and biological applications, the research described here focusses on less familiar cases where diffusion coefficients or other 'shape' functions are defined either in general or poor analytic terms. Of particular interest here is the case of the extension of Richard's equation, which describes the movement of water in a homogeneous unsaturated soil, to cases describing the combined transport of water vapour and heat under a combination of gradients of soil temperature and volumetric water content. The theory was first formulated by Philip and De Vries [13] and DeVries [3], with some practical data provided by Jackson [6]. Such coupled transport is of considerable significance in semi-arid environments where moisture transport often occurs essentially in the water vapour phase. Under these conditions the transport equations, valid in a vertical column of soil, may be written in the form [7]

$$\begin{aligned}\frac{\partial T}{\partial t} &= \frac{\partial}{\partial x}\left[\kappa_T\,(T,\theta)\,\frac{\partial T}{\partial x} + \kappa_\theta\,(T,\theta)\,\frac{\partial \theta}{\partial x}\right]\\ \frac{\partial \theta}{\partial t} &= \frac{\partial}{\partial x}\left[D_T\,(T,\theta)\,\frac{\partial T}{\partial x} + D_\theta\,(T,\theta)\,\frac{\partial \theta}{\partial x}\right]\end{aligned} \tag{1}$$

§ baikov@math.ugatu.ac.ru

‖ gladkov@math.ugatu.ac.ru

¶ rjwiltsh@glamorgan.ac.uk

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where $T(x, t)$ and $\theta(x, t)$ are respectively the soil temperature and volumetric water content at depth x and time t. It is important to realize that extensions which include the coupled diffusion of solute follow in an obvious way.

Of particular interest to the mathematician is the fact that not only are the four diffusion coefficients poorly defined nonlinear functions of T and θ [15], but that there is a recurring requirement on the part of soil scientists for analytic/semi analytic accounts of such transport processes. Indeed Philip [12] stated that one task for the mathematical analyst is to extract essential information from the equations and to characterize solution types of interest. In terms of coupled flow the task may be said to consist firstly, of determining realistic forms for the diffusion equations which give rise to at least partially analytic solutions and, secondly, of providing analytical descriptions of experiments. Although a perturbation analysis of the system (1) was considered by Shepherd and Wiltshire [16] for harmonic soil surface boundary conditions, no explicit analytical forms for the diffusion coefficients were found. Such an analysis is ideally suited to the application of symmetry techniques which can potentially define diffusion functions in such a way as to produce relatively simple solutions which at the same time can be of significant physical interest. This was the case in a one-dimensional point symmetry group analysis of Richard's equation produced by Sposito [19], which was extended to include three-dimensional cases by Edwards and Broadbridge [4] and potential symmetries by Sophocleous [18]. It is interesting to note that these authors found that Lie symmetries did exist for cases in which moisture diffusivities and hydraulic conductivities were either power-law or exponential functions of volumetric moisture content/matric potential. Indeed $D(\theta) = a(b - \theta)^{-2}$ is a functional form of particular significance as shown by Knight and Philip [8] and Broadbridge and White [2]. In the case of coupled diffusion some promising symmetry results were obtained by Wiltshire [21], though the analysis was primarily linear in nature.

It is the aim here to produce an extended symmetry analysis for the cases of linear and nonlinear coupled diffusion and to show how the power-law and exponential diffusivities for the scalar cases can be generalized. Furthermore, particular analytical solutions for coupled diffusion will be given that correspond to impulsive boundary conditions. Finally, the analysis will be discussed in connection with the problem of determining solutions for the nonlinear coupled diffusion of temperature and volumetric moisture content under periodic boundary conditions. Two similarity variables and a perturbation method will be used to generate a solution valid when the matrix of diffusion coefficients is exponentially dependent on the similarity variables.

Our attention will be focused on the one-dimensional nonlinear equation written in the form

$$\boldsymbol{p}(x, t, \boldsymbol{y}, \dot{\boldsymbol{y}}, \boldsymbol{y}', \boldsymbol{y}'') = \frac{\partial \boldsymbol{y}}{\partial t} - \frac{\partial}{\partial x}\left\{\Lambda(\boldsymbol{y})\frac{\partial \boldsymbol{y}}{\partial x}\right\} = 0 \tag{2}$$

where $\boldsymbol{y} = \boldsymbol{y}(x, t)$ with $\boldsymbol{y} = \{y_i\}$, $\quad i = 1, \ldots, n$ and where $\Lambda(\boldsymbol{y})$ is the square matrix of diffusion coefficients, with $\det \Lambda \neq 0$. In many physical situations this matrix is non-singular and sometimes diagonally dominant. The group classification of the scalar case $\boldsymbol{y} = \{y_1\}$, constant Λ, or heat equation is well known [11] and it will be shown that some of the results presented here are similar to the scalar case.

Finally although the focus here will be on applications of coupled diffusion in soil science, such systems are also important in mathematical biology where examples concerning chemotaxis, in which organisms respond positively to a chemical attractant, are also extremely important [9, 17].

The analysis begins in section 2 with the derivation of the determining equations for the classical symmetry groups associated with the system (2) and then continues in section 3 with the presentation of a detailed analysis of the particular groups corresponding to various function forms for $\Lambda(\boldsymbol{y})$. In section 4 a similar analysis is presented for the case when Λ is constant and further a summary of the principal Lie symmetries is given in tabular form. In section 5 this analysis is then used to obtain a symmetry reduction of coupled diffusion corresponding to impulsive boundary conditions, and in section 6 a symmetry reduction linked with perturbation analysis is presented so as to discuss the coupled flow of moisture and heat in a soil.

2. Definition of the classical symmetry groups

A complete account of the method explained in this section may be found in, for example, Olver [10], Bluman and Kumei [1] and Stephani [20].

The infinitesimal generator $\mathcal{X}$ is defined by

$$\mathcal{X} = \xi(x,t,\boldsymbol{y})\frac{\partial}{\partial x} + \eta(x,t,\boldsymbol{y})\frac{\partial}{\partial t} + \boldsymbol{\pi}(x,t,\boldsymbol{y})\cdot\nabla \qquad \nabla = \left\{\frac{\partial}{\partial y_i}\right\} \tag{3}$$

and further the appropriate prolongation operator may be written as

$$\mathcal{X}_E = \xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial t} + \boldsymbol{\pi}\cdot\nabla + \boldsymbol{\pi}_x\cdot\nabla_{y'} + \boldsymbol{\pi}_t\cdot\nabla_{\dot{y}} + \boldsymbol{\pi}_{xx}\cdot\nabla_{y''} \tag{4}$$

where $\boldsymbol{\pi}_x$, $\boldsymbol{\pi}_t$ and $\boldsymbol{\pi}_{xx}$ have familiar forms which are presented in the appendix and where

$$\nabla_{y'} = \left\{\frac{\partial}{\partial y_i'}\right\} \qquad \nabla_{\dot{y}} = \left\{\frac{\partial}{\partial \dot{y}_i}\right\} \qquad \nabla_{y''} = \left\{\frac{\partial}{\partial y_i''}\right\}. \tag{5}$$

The condition for invariance of equation (2) may be found by setting $\mathcal{X}_E\boldsymbol{p}\big|_{(2)} = 0$, with the result that

$$\boldsymbol{\pi}_t = \Lambda\boldsymbol{\pi}_{xx} + \big[(\boldsymbol{\pi}\cdot\nabla)\,\Lambda\big]\boldsymbol{y}'' + \big[(\boldsymbol{y}'\cdot\nabla)\Lambda\big]\boldsymbol{\pi}_x + \big[(\boldsymbol{\pi}_x\cdot\nabla)\Lambda\big]\boldsymbol{y}' + \big[(\boldsymbol{\pi}\cdot\nabla)(\boldsymbol{y}'\cdot\nabla)\Lambda\big]\boldsymbol{y}' \tag{6}$$

which is also presented in an expanded form in the appendix. It may be shown that many of the resulting equations are identically satisfied only if

$$\xi = \xi(x,t) \qquad \eta = \eta(t). \tag{7}$$

The remaining non-trivial determining equations are

$$\dot{\boldsymbol{\pi}} = \Lambda\boldsymbol{\pi}'' \tag{8}$$

$$-\dot{\xi}\boldsymbol{y}' = 2\Lambda(\boldsymbol{y}'\cdot\nabla)\boldsymbol{\pi}' - \xi''\Lambda\boldsymbol{y}' + \big[(\boldsymbol{y}'\cdot\nabla)\Lambda\big]\boldsymbol{\pi}' + \big[(\boldsymbol{\pi}'\cdot\nabla)\Lambda\big]\boldsymbol{y}' \tag{9}$$

$$(\Lambda\boldsymbol{y}''\cdot\nabla)\boldsymbol{\pi} - \dot{\eta}\Lambda\boldsymbol{y}'' = \Lambda\big[(\boldsymbol{y}''\cdot\nabla)\boldsymbol{\pi}\big] - 2\xi'\Lambda\boldsymbol{y}'' + \big[(\boldsymbol{\pi}\cdot\nabla)\Lambda\big]\boldsymbol{y}'' \tag{10}$$

$$(\Lambda\boldsymbol{y}'\cdot\nabla)(\boldsymbol{y}'\cdot\nabla)\boldsymbol{\pi} = 0. \tag{11}$$

By inspection of the determining equations (8)–(11) the *principal Lie algebra* is the three-dimensional space of infinitesimal symmetries of equation (2) spanned by the operators

$$\mathcal{X}_1 = \frac{\partial}{\partial x} \qquad \mathcal{X}_2 = \frac{\partial}{\partial t} \qquad \mathcal{X}_3 = x\frac{\partial}{\partial x} + 2t\frac{\partial}{\partial t}. \tag{12}$$

3. Solution of the determining equations: $\Lambda \neq$ constant

3.1. Λ is an arbitrary matrix and depends on $y_1, y_2, \ldots, y_k$

If Λ depends only on k variables, so that $\Lambda = \Lambda(y_1, y_2, \ldots, y_k)$, $0 < k < n$, then the Lie algebra can be extended to include

$$\mathcal{X}_{3+i} = \frac{\partial}{\partial y_{k+i}} \qquad i = 1, 2, \ldots, (n-k). \tag{13}$$

3.2. Basic simplification

For particular cases of Λ, the system (8)–(11) can be satisfied without each of the coefficients being zero. In these cases, extensions to the algebra (12) are possible. In this paper the case

$$\boldsymbol{\pi} = m(x,t)\boldsymbol{\mu}(\boldsymbol{y}) \tag{14}$$

will be considered where $m(x,t)$ is a scalar function and $\boldsymbol{\mu}$ is a vector function of $\boldsymbol{y}$.

For the classification of cases when the algebra (12) may be extended, first note that the determining equations may also be written as

$$a\boldsymbol{\mu} = b\Lambda\boldsymbol{\mu} \tag{15}$$

$$-c\boldsymbol{y}' = 2d\Lambda(\boldsymbol{y}'\cdot\nabla)\boldsymbol{\mu} - e\Lambda\boldsymbol{y}' + d\big[(\boldsymbol{y}'\cdot\nabla)\Lambda\big]\boldsymbol{\mu} + d\big[(\boldsymbol{\mu}\cdot\nabla)\Lambda\big]\boldsymbol{y}' \tag{16}$$

$$(\Lambda\boldsymbol{y}'\cdot\nabla)\boldsymbol{\mu} + \sigma\Lambda\boldsymbol{y}' = \Lambda\big[(\boldsymbol{y}'\cdot\nabla)\boldsymbol{\mu}\big] + (\boldsymbol{\mu}\cdot\nabla)\Lambda\boldsymbol{y}' \tag{17}$$

$$(\Lambda\boldsymbol{y}'\cdot\nabla)(\boldsymbol{y}'\cdot\nabla)\boldsymbol{\mu} = 0 \tag{18}$$

where a, b, c, d, e and σ are constant coefficients ($\det\Lambda \neq 0$). Indeed, since Λ depends only on $\boldsymbol{y}$ then (8) to (11) can hold only when all of the coefficients vanish identically or are proportional, with constant coefficients, to the functions $m(x,t) \neq 0$, $\lambda_1(x,t) \neq 0$ and $\lambda_2(x,t) \neq 0$. Namely

$$\dot{m}(x,t) = a\lambda_1(x,t) \qquad m''(x,t) = b\lambda_1(x,t) \tag{19}$$

$$\dot{\xi} = c\lambda_2(x,t) \qquad m'(x,t) = d\lambda_2(x,t) \qquad \xi'' = e\lambda_2(x,t) \tag{20}$$

$$(2\xi' - \dot{\eta}) = \sigma m(x,t). \tag{21}$$

It is easy to see from (20) and (21) that

$$2e = \sigma d. \tag{22}$$

Note that the system (15)–(18) is not compatible whenever $\boldsymbol{\mu}$ is an arbitrary vector function. Thus $\boldsymbol{\mu}$ may be found by writing

$$\boldsymbol{\mu}(\boldsymbol{y}) = \boldsymbol{\mu}_0(\boldsymbol{y}) + \sum_{i=1}^{l} \tilde{c}_i\boldsymbol{\mu}_i(\boldsymbol{y}) \qquad l \in \aleph \tag{23}$$

where $\tilde{c}_1, \tilde{c}_2, \ldots, \tilde{c}_l$ are arbitrary constants and Λ does not depend on $\tilde{c}_i$. Here, $\boldsymbol{\mu}_0$ and $\boldsymbol{\mu}_i$ ($i = 1, \ldots, l$) are linear independent solutions of the system (15)–(18) for a given matrix Λ.

On substitution of (23) for $\boldsymbol{\mu}(\boldsymbol{y})$ in the system (15)–(18) and by identifying coefficients of $\tilde{c}_i$, then the following system may be obtained:

$$a\boldsymbol{\mu}_0 = b\Lambda\boldsymbol{\mu}_0 \tag{24}$$

$$-c\boldsymbol{y}' = 2d\Lambda(\boldsymbol{y}'\cdot\nabla)\boldsymbol{\mu}_0 - \tfrac{1}{2}\sigma d\Lambda\boldsymbol{y}' + d\big[(\boldsymbol{y}'\cdot\nabla)\Lambda\big]\boldsymbol{\mu}_0 + d\big[(\boldsymbol{\mu}_0\cdot\nabla)\Lambda\big]\boldsymbol{y}' \tag{25}$$

$$(\Lambda\boldsymbol{y}'\cdot\nabla)\boldsymbol{\mu}_0 + \sigma\Lambda\boldsymbol{y}' = \Lambda\big[(\boldsymbol{y}'\cdot\nabla)\boldsymbol{\mu}_0\big] + (\boldsymbol{\mu}_0\cdot\nabla)\Lambda\boldsymbol{y}' \tag{26}$$

$$(\Lambda\boldsymbol{y}'\cdot\nabla)(\boldsymbol{y}'\cdot\nabla)\boldsymbol{\mu}_0 = 0 \tag{27}$$

$$a\boldsymbol{\mu}_i = b\Lambda\boldsymbol{\mu}_i \tag{28}$$

$$d\left[2\Lambda(\boldsymbol{y}'\cdot\nabla)\boldsymbol{\mu}_i + \big[(\boldsymbol{y}'\cdot\nabla)\Lambda\big]\boldsymbol{\mu}_i + \big[(\boldsymbol{\mu}_i\cdot\nabla)\Lambda\big]\boldsymbol{y}'\right] = 0 \tag{29}$$

$$(\Lambda\boldsymbol{y}'\cdot\nabla)\boldsymbol{\mu}_i = \Lambda\big[(\boldsymbol{y}'\cdot\nabla)\boldsymbol{\mu}_i\big] + (\boldsymbol{\mu}_i\cdot\nabla)\Lambda\boldsymbol{y}' \tag{30}$$

$$(\Lambda\boldsymbol{y}'\cdot\nabla)(\boldsymbol{y}'\cdot\nabla)\boldsymbol{\mu}_i = 0. \tag{31}$$

Since $\det\Lambda \neq 0$, it follows from (24) and (28) that the coefficients a and b can only be zero in simultaneous cases. Hence, further analysis of (24)–(31) will be continued in two separate classes.

3.3. *The class $a \neq 0,\ b \neq 0$*

In this class equations (24) and (28) can be considered as an eigenvalue problem, where $\lambda_\Lambda = a/b$ is the eigenvalue and $\boldsymbol{\mu}$ is the eigenvector of Λ. If Λ is such that there are several eigenvectors corresponding to the same eigenvalue λ_Λ, then in (23) $l \leqslant n-1$. In this case it may be shown that

$$\sigma = 0 \qquad e = 0 \qquad cm''' = 0. \tag{32}$$

3.3.1. Subclass $c \neq 0$. The matrix Λ and vectors $\boldsymbol{\mu}_0$ and $\boldsymbol{\mu}_i$ $(i = 1, \ldots, l)$ are defined from the system (24)–(31) with

$$\sigma = 0 \qquad c = d. \tag{33}$$

When the system (24)–(31) is compatible, then

$$\begin{aligned}
&\mathcal{X}_4 = t^2\frac{\partial}{\partial t} + tx\frac{\partial}{\partial x} + \left(\frac{x^2}{2} + \lambda_\Lambda t\right)\boldsymbol{\mu}_0\cdot\nabla \qquad \mathcal{X}_5 = t\frac{\partial}{\partial x} + x\boldsymbol{\mu}_0\cdot\nabla\\
&\mathcal{X}_{6+i} = \boldsymbol{\mu}_i\cdot\nabla \qquad \mathcal{X}_{6+l+i} = x\boldsymbol{\mu}_i\cdot\nabla\\
&\mathcal{X}_{6+2l+i} = \left(\tfrac{1}{2}x^2 + \lambda_\Lambda t\right)x\boldsymbol{\mu}_i\cdot\nabla \qquad i = 1, \ldots, l.
\end{aligned} \tag{34}$$

Remark. The system (24)–(31) is incompatible when $\boldsymbol{\mu}_0 = N\boldsymbol{y} + \boldsymbol{k}$, where N is a $n \times n$ constant matrix and $\boldsymbol{k}$ is a constant vector.

3.3.2. Subclass $c = 0$. The matrix Λ and vectors $\boldsymbol{\mu}_0$, $\boldsymbol{\mu}_i$ $(i = 1, \ldots, l)$ are defined by (24)–(31) with $\sigma = 0,\ c = 0$. When the system (24)–(31) is compatible, then

$$\mathcal{X}_{4+i} = m(x,t)\boldsymbol{\mu}_i\cdot\nabla \qquad i = 0, \ldots, l \tag{35}$$

where m satisfies $\dot{m} = \lambda_\Lambda m''$. So, when $\boldsymbol{\mu}_0$ and $\boldsymbol{\mu}_i$ $(i = 1, \ldots, l)$ are constant vectors, then $\boldsymbol{\mu}_i \equiv \boldsymbol{h}$ $(i = 0, \ldots, l)$ where $\boldsymbol{h}$ is a constant vector and Λ may be found as a solution of the system (24)–(31). Hence

$$\Lambda = \Psi_1(\boldsymbol{y}) = \Psi_1\left(\frac{y_1}{h_1} - \frac{y_2}{h_2}, \frac{y_1}{h_1} - \frac{y_3}{h_3}, \ldots, \frac{y_1}{h_1} - \frac{y_n}{h_n}\right) \qquad \lambda_\Lambda h_i = \sum_j h_j \Psi_{ij} \tag{36}$$

where $i, j = 1, \ldots, n$. The additional symmetry operator is

$$\mathcal{X}_\infty = m(x,t)\boldsymbol{h}\cdot\nabla. \tag{37}$$

3.4. The class $a = 0$, $b = 0$

3.4.1. Subclass 1. $d \neq 0$. The matrix Λ and vectors $\boldsymbol{\mu}_0$ and $\boldsymbol{\mu}_i$ $(i = 1, \ldots, l)$ are defined from (25)–(27), (29)–(31). In the cases when the system is compatible, the additional symmetry operators are

$$\begin{gathered}\mathcal{X}_4 = -\sigma t\frac{\partial}{\partial t} + \boldsymbol{\mu}_0\cdot\nabla \qquad \mathcal{X}_5 = \left(\frac{\sigma}{4}x^2 + \frac{c}{d}t\right)\frac{\partial}{\partial x} + x\boldsymbol{\mu}_0\cdot\nabla \\ \mathcal{X}_{5+i} = \boldsymbol{\mu}_i\cdot\nabla \qquad \mathcal{X}_{5+l+i} = x\boldsymbol{\mu}_i\cdot\nabla \qquad i = 1, \ldots, l.\end{gathered} \tag{38}$$

To find the explicit form for $\Lambda(\boldsymbol{y})$ the particular case, when $\boldsymbol{\mu}_0 = N\boldsymbol{y} + \boldsymbol{k}$ will now be considered, where N is an $n \times n$ constant matrix and $\boldsymbol{k}$ is a constant vector.

Case (a). The system $N\boldsymbol{y}+\boldsymbol{k} = \boldsymbol{0}$ has the solution $\boldsymbol{y} = \boldsymbol{c}$. Since $\det N \neq 0$ then eigenvalues $\lambda \neq 0$ and regular transformations exist, such that

$$\boldsymbol{y} = B\boldsymbol{v} + \boldsymbol{c} \qquad \nabla_v = B^{\mathrm{T}}\nabla \qquad N = \lambda I + M \qquad B^{-1}MB = 0 \qquad \Lambda = B\overline{\Lambda}B^{-1}. \tag{39}$$

Subcase (i). It may be shown that in this case

$$c = 0 \qquad \sigma = -\tfrac{4}{3}\lambda \tag{40}$$

and the matrix $\overline{\Lambda}$ has the form

$$\begin{gathered}\overline{\Lambda} = v_1^{-4/3}\Psi_2(\boldsymbol{v}) \\ \Psi_2(\boldsymbol{v}) = \begin{pmatrix} -\frac{1}{3}\Psi_1 - I_2\Psi_{1,I_2} - \ldots - I_n\Psi_{1,I_n} & \Psi_{1,I_2} & \cdots & \Psi_{1,I_n} \\ -\frac{1}{3}\Psi_2 - I_2\Psi_{2,I_2} - \cdots - I_n\Psi_{2,I_n} & \Psi_{2,I_2} & \ldots & \Psi_{2,I_n} \\ \vdots & \vdots & \ddots & \vdots \\ -\frac{1}{3}\Psi_n - I_2\Psi_{n,I_2} - \cdots - I_n\Psi_{n,I_n} & \Psi_{n,I_2} & \cdots & \Psi_{n,I_n} \end{pmatrix}\end{gathered} \tag{41}$$

where

$$\Psi_l = \Psi_l(I_2, \ldots, I_n) \qquad \Psi_{l,I_j} = \frac{\partial\Psi_l}{\partial I_j} \qquad I_j = \frac{v_j}{v_1} \quad j = 2, \ldots, n \quad l = 1, \ldots, n. \tag{42}$$

The generators $\mathcal{X}_4$ and $\mathcal{X}_5$ from (38) are

$$\mathcal{X}_4 = \frac{4}{3}t\frac{\partial}{\partial t} + \boldsymbol{v}\cdot\nabla_v \qquad \mathcal{X}_5 = -\frac{1}{3}x^2\frac{\partial}{\partial x} + x\boldsymbol{v}\cdot\nabla_v. \tag{43}$$

Subcase (ii). In this case

$$c = 0 \qquad \sigma = -2\lambda \qquad \overline{\Lambda}\boldsymbol{v} = \boldsymbol{0} \tag{44}$$

and the matrix has the form

$$\overline{\Lambda} = v_1^{-2}\Psi_3(\boldsymbol{v}) \qquad \Psi_3(\boldsymbol{v}) = \Psi_3\left(\frac{v_2}{v_1}, \ldots, \frac{v_n}{v_1}\right). \tag{45}$$

The generators $\mathcal{X}_4$ and $\mathcal{X}_5$ from (38) are

$$\mathcal{X}_4 = 2t\frac{\partial}{\partial t} + \boldsymbol{v}\cdot\nabla_v \qquad \mathcal{X}_5 = -\frac{1}{2}x^2\frac{\partial}{\partial x} + x\boldsymbol{v}\cdot\nabla_v. \tag{46}$$

Case (b). The system $N\boldsymbol{y} + \boldsymbol{k} = \boldsymbol{0}$ is incompatible. Consider

$$\boldsymbol{y} = A\boldsymbol{z} \qquad A^{-1}NA = 0 \qquad \nabla_z = A^{\mathrm{T}}\nabla \qquad \overline{\Lambda} = A^{-1}\Lambda A \qquad \boldsymbol{h} = A^{-1}\boldsymbol{k}.$$

In this case

$$\overline{\Lambda}(\boldsymbol{z}) = \Psi\left(\frac{z_1}{h_1} - \frac{z_2}{h_2}, \frac{z_1}{h_1} - \frac{z_3}{h_3}, \ldots, \frac{z_1}{h_1} - \frac{z_n}{h_n}\right) \tag{47}$$

where $\boldsymbol{h} = \{h_i\}$, $h_1 \neq 0$ satisfying $\overline{\Lambda}\boldsymbol{h} = \boldsymbol{l}$, where $\boldsymbol{l}$ is a constant vector. The generators $\mathcal{X}_4$ and $\mathcal{X}_5$ from (38) are

$$\mathcal{X}_4 = \boldsymbol{h}\cdot\nabla_z \qquad \mathcal{X}_5 = x\boldsymbol{h}\cdot\nabla_z. \tag{48}$$

3.4.2. Subclass 2. $d = 0$. It is clear from equation (32) that $c = 0$. The matrix Λ and vectors $\boldsymbol{\mu}_0$ and $\boldsymbol{\mu}_i$ $(i = 1, \ldots, l)$ are defined by equations (25)–(27), (29)–(31) with $d = 0$ and $c = 0$. In the cases when this system is compatible, the additional symmetry operators have the following forms:

$$\mathcal{X}_4 = -\sigma t\frac{\partial}{\partial t} + \boldsymbol{\mu}_0\cdot\nabla \qquad \mathcal{X}_{4+i} = \boldsymbol{\mu}_i\cdot\nabla \qquad i = 1, \ldots, l. \tag{49}$$

Consider some particular cases of the compatibility of this system.

Case (a). $\boldsymbol{\mu}_0 = N\boldsymbol{y} + \boldsymbol{k}$. In this case, (27) is an identity and the solution of (26) may be considered in two separate subcases similar to the case $d \neq 0$.

Subcase (i). The system $N\boldsymbol{y}+\boldsymbol{k} = \boldsymbol{0}$ has the solution $\boldsymbol{y} = \boldsymbol{c}$. Using (39) then (24)–(31), (49) become

$$\begin{aligned} &\overline{\Lambda}(\boldsymbol{v}) = v_1^{\sigma/\lambda}\Psi_3(\boldsymbol{v}) \qquad \Psi_3(\boldsymbol{v}) = \Psi_3\left(\frac{v_2}{v_1}, \frac{v_3}{v_1}, \ldots, \frac{v_n}{v_1}\right) \\ &\mathcal{X}_4 = -\sigma t\frac{\partial}{\partial t} + \lambda\boldsymbol{v}\cdot\nabla_v \end{aligned} \tag{50}$$

where $\Psi(\boldsymbol{y})$ is an $n \times n$ matrix.

Subcase (ii): The system $N\boldsymbol{y} + \boldsymbol{k} = \boldsymbol{0}$ is incompatible. In this case

$$\overline{\Lambda}(\boldsymbol{z}) = \exp\left(\frac{\sigma}{h_1}z_1\right)\Psi_4(\boldsymbol{z})$$
$$\Psi(\boldsymbol{z}) = \Psi_4\left(\frac{z_1}{h_1} - \frac{z_2}{h_2}, \frac{z_1}{h_1} - \frac{z_3}{h_3}, \ldots, \frac{z_1}{h_1} - \frac{z_n}{h_n}\right) \tag{51}$$

and the particular form of $\mathcal{X}_4$ from (49) is

$$\mathcal{X}_4 = -\sigma t \frac{\partial}{\partial t} + \boldsymbol{h}\boldsymbol{\cdot}\nabla_z. \tag{52}$$

Case (b). Λ is an upper-triangular matrix and $n = 2$. It may be shown that the system (24)–(31) is compatible, when $\boldsymbol{\mu}_0$ is nonlinear. In particular

$$\boldsymbol{\mu}_0 = \begin{pmatrix} y_1\alpha(y_2) + \beta(y_2) \\ \gamma_1 y_2 + \gamma_2 \end{pmatrix} \qquad \Lambda = (\gamma_1 y_2 + \gamma_2)^\sigma \psi(J)\begin{pmatrix} 1 & \tau \\ 0 & -1 \end{pmatrix} \tag{53}$$

where

$$\tau = \frac{1}{\alpha'}(y_1\alpha'' + \beta'') \qquad (\gamma_1 y_2 + \gamma_2)\frac{\alpha''}{\alpha'} + 2\alpha + \gamma_3 = 0 \tag{54}$$

$$\beta''' + \beta''\left(-\frac{\alpha''}{\alpha'} + \gamma_1 + \alpha\right) + 2\beta'\alpha' + \beta\alpha'' = 0. \tag{55}$$

The function ψ is arbitrary and J is an invariant of the equation

$$\boldsymbol{\mu}_0^1\frac{\partial}{\partial y_1}J + \boldsymbol{\mu}_0^2\frac{\partial}{\partial y_2}J = 0 \tag{56}$$

and $\gamma_1,\ \gamma_2,\ \gamma_3$ are arbitrary constants.

4. Solution of the determining equations when $\Lambda = \Lambda^{[0]}$ is constant

In the particular case when $\Lambda = \Lambda^{[0]}$ is constant the determining equations (8) to (11) may be solved explicitly as follows. It follows from (11) that $\Lambda^{[0]}(\boldsymbol{y}'\boldsymbol{\cdot}\nabla)^2\boldsymbol{\pi} = 0$ and since $\det\Lambda^{[0]} \neq 0$, then

$$\boldsymbol{\pi} = M(x,t)\boldsymbol{y} + \beta(x,t) \tag{57}$$

where M is a $n \times n$ matrix and β is a vector, such that equations (8)–(11) become

$$\dot{M} = \Lambda^{[0]}M'' \qquad \dot{\boldsymbol{\beta}} = \Lambda\boldsymbol{\beta}'' \tag{58}$$

$$-\dot{\xi}I = \Lambda^{[0]}M' + M'\Lambda^{[0]} + \Lambda^{[0]}\xi'' \tag{59}$$

$$(2\xi' - \dot{\eta})\Lambda^{[0]} + M\Lambda^{[0]} - \Lambda^{[0]}M = 0. \tag{60}$$

On differentiation of (59) with respect to x and with the aid of (60) differentiated twice with respect to x, it may be shown that

$$-\Lambda^{[0]}\dot{\xi}' = \tfrac{3}{2}\Lambda^{[0]}\dot{M} + \tfrac{1}{2}\dot{M}\Lambda^{[0]}. \tag{61}$$

In addition, since $\det\Lambda^{[0]} \neq 0$, from equation (60) we have $\ddot{\eta}I = -4\dot{M} \Rightarrow \dot{M} = \dot{h}(t)I$. Hence

$$M(x,t) = h(t)I + G(x) \qquad \dot{\eta} = -4h(t) + a_0 \tag{62}$$

where a_0 is a constant. It follows on substitution of (62) in (61) that

$$\xi' = -2h(t) + g(x). \tag{63}$$

If the relationships (62) to (63) are now successively substituted in (58), (60) and then (58) the exact form of $\xi(x,t)$, $\eta(t)$ and $M(x,t)$ found. In this way it may be shown that in addition to (12) the infinitesimal generators have the form

$$\mathcal{X}_4 = 2t\frac{\partial}{\partial x} - x\Lambda^{[0]}\boldsymbol{y}\cdot\nabla \tag{64}$$

$$\mathcal{X}_5 = 4tx\frac{\partial}{\partial x} + 4t^2\frac{\partial}{\partial t} - (x^2\Lambda^{[0]} + 2tI)\boldsymbol{y}\cdot\nabla \tag{65}$$

$$\mathcal{X}_6 = A\boldsymbol{y}\cdot\nabla \qquad \mathcal{X}_7 = -c_6x\frac{\partial}{\partial x} \qquad \mathcal{X}_\infty = \boldsymbol{\beta}\cdot\nabla \tag{66}$$

where c_6 and A are related through

$$A\Lambda^{[0]} - \Lambda^{[0]}A = 2c_6\Lambda^{[0]}. \tag{67}$$

If $\det\Lambda^{[0]} \neq 0$ then equation (67) has a solution if and only if $c_6 = 0$. A summary of the results is presented in table 1.

Table 1. Principal examples of classical symmetry. $\dot{\boldsymbol{\beta}} = \Lambda^{[0]}\boldsymbol{\beta}''$, $\dot{m} = (a/b)m''$ and $\Psi_i(\boldsymbol{y})$ are defined in section 3.

$\Lambda(\boldsymbol{y})$	ξ	η	$\boldsymbol{\pi}$
$\Lambda(\boldsymbol{y})$ $\boldsymbol{y} = \{y_1 \ldots y_n\}$	$c_1x + c_2$	$2c_1t + c_3$	$\mathbf{0}$
$\Lambda(\boldsymbol{y})$ $\boldsymbol{y} = \{y_1 \ldots y_m\}$ $m < n$	$c_1x + c_2$	$2c_1t + c_3$	$\boldsymbol{h} = \{h_1, \ldots, h_m\}$
$\Lambda^{[0]}$	$(c_1 - c_6)x + c_4tx + c_5t + c_2$	$c_4t^2 + 2c_1t + c_3$	$-\left\{\frac{1}{4}c_4t + \frac{1}{2}\Lambda_0^{-1}c_4x^2\frac{1}{2}\Lambda_0^{-1}c_5x + A\right\}\boldsymbol{y} + \boldsymbol{\beta}$
$\Psi_1(\boldsymbol{y})$	$c_1x + c_2$	$2c_1t + c_3$	$m\boldsymbol{h}$
$(y_1 + k_1)^{-4/3}\Psi_2(\boldsymbol{y})$	$-\frac{1}{3}c_5x^2 + c_1x + c_2$	$(2c_1 + \frac{4}{3}c_4)t + c_3$	$(c_5x + c_4)(\boldsymbol{y} + \boldsymbol{k})$
$(y_1 + k_1)^{-2}\Psi_3(\boldsymbol{y})$	$-\frac{1}{2}c_5x^2 + c_1x + c_2$	$(2c_1 + 2c_4)t + c_3$	$(c_5x + c_4)(\boldsymbol{y} + \boldsymbol{k})$
$(y_1 + k_1)^{\sigma/\lambda}\Psi_3(\boldsymbol{y})$	$c_1x + c_2$	$(2c_1 - \sigma c_4)t + c_3$	$c_4\lambda(\boldsymbol{y} + \boldsymbol{k})$
$e^{\sigma(y_1+k_1)/\lambda}\Psi_4(\boldsymbol{y})$	$c_1x + c_2$	$(2c_1 - \sigma c_4)t + c_3$	$c_4\boldsymbol{h}$

5. Self-similar solution for impulsive boundary conditions

In this section consideration is given to symmetry reduction for the case of impulsive boundary conditions. In particular, suppose that $\boldsymbol{y} = [y_1, y_2]^{\mathrm{T}}$ and

$$\frac{\partial \boldsymbol{y}}{\partial t} = \frac{\partial}{\partial x}\left\{y_1^k\boldsymbol{\Psi}\left(\frac{y_2}{y_1}\right)\frac{\partial \boldsymbol{y}}{\partial x}\right\} \qquad y_1, y_2 \geqslant 0 \quad t > t_0 \tag{68}$$

$$\boldsymbol{y}(x, t_0) = a\delta(x - x_0)\boldsymbol{y}_0 \tag{69}$$

where $\delta(x - x_0)$ is the Dirac measure at x_0.

According to the invariance principle, it is necessary to find the subalgebra of the Lie algebra spanned by

$$\mathcal{X}_1 = \frac{\partial}{\partial t} \qquad \mathcal{X}_2 = \frac{\partial}{\partial x} \qquad \mathcal{X}_3 = 2t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x} \tag{70}$$

$$\mathcal{X}_4 = -kt\frac{\partial}{\partial t} + y_1\frac{\partial}{\partial y_1} + y_2\frac{\partial}{\partial y_2} \tag{71}$$

which leaves the *initial manifold*, the line $t = t_0$, invariant and which also conserves the *initial conditions* at $t = t_0$ given by $x = x_0$ and by equation (69). This subalgebra is the two-dimensional algebra spanned by

$$\mathcal{Y}_1 = 2(t - t_0)\frac{\partial}{\partial t} + (x - x_0)\frac{\partial}{\partial x} \tag{72}$$

$$\mathcal{Y}_2 = -k(t - t_0)\frac{\partial}{\partial t} + y_1\frac{\partial}{\partial y_1} + y_2\frac{\partial}{\partial y_2}. \tag{73}$$

Hence, in summary it is necessary to consider the generator

$$\mathcal{Y} = \frac{1}{k+2}(\mathcal{Y}_1 - \mathcal{Y}_2) = (t - t_0)\frac{\partial}{\partial t} + \frac{1}{k+2}(x - x_0)\frac{\partial}{\partial x} - \frac{1}{k+2}\left(y_1\frac{\partial}{\partial y_1} + y_2\frac{\partial}{\partial y_2}\right). \tag{74}$$

Using the method of characteristics the similary ansatz is found to be

$$\boldsymbol{y}(x, t) = (t - t_0)^{-1/(k+2)}\varphi(\omega) \qquad \omega = \frac{x - x_0}{(t - t_0)^{1/(k+2)}} \qquad k + 2 \neq 0 \tag{75}$$

and so the diffusion equation reduces to

$$-\frac{1}{k+2}\omega\varphi = \varphi_1^k\left[\boldsymbol{\Psi}\frac{\mathrm{d}\varphi}{\mathrm{d}\omega}\right] + \varphi_0 \tag{76}$$

where φ_0 may be taken as zero when φ remains bounded for all ω.

5.1. *General solution*

In this case equation (76) may be rewritten in the following form:

$$\frac{\mathrm{d}\varphi_1}{\mathrm{d}\omega} = \frac{1}{D(k+2)}\omega\left[\Psi_{21}\frac{\varphi_1}{\varphi_1^k} - \Psi_{11}\frac{\varphi_2}{\varphi_1^k}\right] \tag{77}$$

$$\frac{\mathrm{d}\varphi_2}{\mathrm{d}\omega} = \frac{1}{D(k+2)}\omega\left[\Psi_{22}\frac{\varphi_1}{\varphi_1^k} - \Psi_{12}\frac{\varphi_2}{\varphi_1^k}\right] \tag{78}$$

where

$$D = \Psi_{11}\Psi_{22} - \Psi_{12}\Psi_{21}. \tag{79}$$

Using the ratio of these equations it is easy to establish the implicit solutions for φ_1 and φ_2 in the terms as follows:

$$\varphi_1 = w(\omega)\varphi_2 \qquad \varphi_2 = C\exp\left(\int\frac{\mathrm{d}w}{\Phi(w)}\right) \tag{80}$$

$$\frac{1}{2(k+2)}(\omega_0^2 - \omega^2) = \int\frac{\mathrm{d}w}{M(w)} = F(w) \tag{81}$$

so that

$$w = F^{-1}\left(\frac{1}{2(k+2)}(\omega_0^2 - \omega^2)\right) \tag{82}$$

where

$$\Phi(w) = \frac{\Psi_{22}w - \Psi_{12}}{\Psi_{21}w - \Psi_{11}} - w$$
$$M(w) = \frac{\Phi(w)}{D}\left[\Psi_{21}w - \Psi_{11}\right]C^{-k}\exp\left(-k\int\frac{\mathrm{d}w}{\Phi(w)}\right)w^{-k}. \tag{83}$$

In addition, C and ω_0 are constants which are determined from the *initial conditions*.

5.2. A particular example

In the case when

$$\boldsymbol{\Psi} = \gamma\left(\frac{\varphi_2}{\varphi_1}\right)\boldsymbol{\Psi}_0 \tag{84}$$

where $\gamma\left(\varphi_2/\varphi_1\right)$ is an arbitrary function, the equation (76) may be solved by supposing that the matrix $\boldsymbol{\Psi}_0$ has two distinct eigenvalues with

$$\boldsymbol{\Psi}_0\boldsymbol{m} = \lambda_1\boldsymbol{m} \qquad \boldsymbol{\Psi}_0\boldsymbol{n} = \lambda_2\boldsymbol{n} \tag{85}$$

and by supposing that

$$\varphi(\omega) = \alpha(\omega)\boldsymbol{m} + \beta(\omega)\boldsymbol{n}. \tag{86}$$

In this way it may be shown that

$$-\frac{1}{k+2}\omega\alpha = \lambda_1\left(\alpha(\omega)m_1 + \beta(\omega)n_1\right)^k\gamma\left(\frac{\alpha(\omega)m_2 + \beta(\omega)n_2}{\alpha(\omega)m_1 + \beta(\omega)n_1}\right)\frac{\mathrm{d}\alpha}{\mathrm{d}\omega} \tag{87}$$

$$-\frac{1}{k+2}\omega\beta = \lambda_2\left(\alpha(\omega)m_1 + \beta(\omega)n_1\right)^k\gamma\left(\frac{\alpha(\omega)m_2 + \beta(\omega)n_2}{\alpha(\omega)m_1 + \beta(\omega)n_1}\right)\frac{\mathrm{d}\beta}{\mathrm{d}\omega}. \tag{88}$$

With the help of the ratio of these equations it is easy to establish the implicit solutions for α and β in the terms as follows:

$$\alpha(\beta) = \alpha_0\beta^{\lambda_2/\lambda_1} \tag{89}$$

$$\frac{1}{2(k+2)}(\omega_0^2 - \omega^2) = \lambda_2\int\left(\alpha(\omega)m_1 + \beta(\omega)n_1\right)^k\gamma\left(\frac{\alpha(\omega)m_2 + \beta(\omega)n_2}{\alpha(\omega)m_1 + \beta(\omega)n_1}\right)\frac{\mathrm{d}\beta}{\beta} = N(\beta) \tag{90}$$

$$\beta(\omega) = N^{-1}\left(\frac{1}{2(k+2)}(\omega_0^2 - \omega^2)\right). \tag{91}$$

Therefore equations (75), (86), (89) and (91) give self-similar solution of (68), (69).

In particular, when $m_i, n_i \geqslant 0$ and since $\beta \geqslant 0$ we have

$$\beta = 0 \qquad |\omega| > \omega_0 \tag{92}$$

and β is determined from equation (91) for $|\omega| \leqslant \omega_0$.

Moreover, without loss generality it may be supposed that $\lambda_1 > \lambda_2$ and it is clear that when $\omega \sim \omega_0$

$$\beta = \left[\frac{k}{2(k+2)\lambda_2 n_1^k\gamma\left(n_2/n_1\right)}(\omega_0^2 - \omega^2)\right]^{1/k} + \mathrm{O}\left((\omega_0^2 - \omega^2)^{1/k+p}\right)$$
$$k > 0 \quad p \geqslant 0. \tag{93}$$

In the case when $k = 0$ and $\gamma \equiv 1$ then (89) and (90) can be integrated and combined with (86) to give

$$\varphi(\omega) = \alpha_0 \exp\left(-\frac{\omega^2}{4\lambda_1}\right)\boldsymbol{m} + \beta_0 \exp\left(-\frac{\omega^2}{4\lambda_2}\right)\boldsymbol{n}. \tag{94}$$

The constant and can be found from the conservation relationship

$$\int_{-\infty}^{\infty} \boldsymbol{y}(x, t_0)\, \mathrm{d}x = \int_{-\infty}^{\infty} \varphi(\omega)\, \mathrm{d}\omega = a\boldsymbol{y}_0. \tag{95}$$

It should be noted that for practical situations the solutions presented in this section may be applied in ideal situations when the solute and moisture distributions have finite integrals, as is the case when spillages occur. In the description of the coupled diffusion of heat and moisture, subject to diurnal or annual variation, such solutions are not valid, and an analysis of the type to be presented in section 6 becomes more appropriate.

6. Application to coupled flow in porous media

6.1. Constant $\Lambda^{[0]}$

Of particular physical interest is the case of the coupled diffusion of temperature, $y_1(x, t)$, and volumetric water content, $y_2(x, t)$, in which the diffusion processes are subject to periodic diurnal or annual boundary conditions. Indeed, in experiments conducted *in situ* by Jackson [6] for Adelanto loam at a site in Arizona, and by Rose [14] for a sandy loam in the Northern Territory, Australia, the elements of the matrix of diffusion coefficients $\Lambda(\boldsymbol{y})$ varied about the constant matrix values of $\Lambda^{[0]}$ such that

$$\Lambda_J^{[0]} = \begin{bmatrix} 2.23 \times 10^{-3} & 4.18 \times 10^{-3} \\ 5.31 \times 10^{-8} & 5.07 \times 10^{-5} \end{bmatrix} \qquad \Lambda_R^{[0]} = \begin{bmatrix} 2.16 \times 10^{-3} & 8.56 \times 10^{-5} \\ 6.21 \times 10^{-8} & 5.11 \times 10^{-5} \end{bmatrix} \tag{96}$$

calculated for the respective Jackson and Rose experiments using their CGS units.

Consider the case when $\Lambda(\boldsymbol{y}) = \Lambda^{[0]}$ is a constant. It may be seen from equations (64), (66) that equation (2) admits the following subalgebra spanned by:

$$\mathcal{Y}_1 = A_1\boldsymbol{y}\cdot\nabla - \alpha_1\frac{\partial}{\partial x} + \omega\frac{\partial}{\partial t} \qquad \mathcal{Y}_2 = A_2\boldsymbol{y}\cdot\nabla - \alpha_2\frac{\partial}{\partial x} + \omega\frac{\partial}{\partial t} \tag{97}$$

$$A_i = \Lambda^{[0]}BC_i\left(\Lambda^{[0]}\right)^{-1} \qquad B = (\boldsymbol{n}_1, \boldsymbol{n}_2) \quad i = 1, 2 \tag{98}$$

$$C_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \qquad C_2 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \tag{99}$$

where $\boldsymbol{n}_1, \boldsymbol{n}_2$ are the eigenvectors of $\Lambda^{[0]}$ with distinct eigenvalues λ_1 and λ_2.

The transformation

$$\boldsymbol{y} = B\boldsymbol{v} \tag{100}$$

gives rise to

$$\mathcal{Z}_1 = v_1\frac{\partial}{\partial v_1} - \alpha_1\frac{\partial}{\partial x} - \omega\frac{\partial}{\partial t} \qquad \mathcal{Z}_2 = v_2\frac{\partial}{\partial v_2} - \alpha_2\frac{\partial}{\partial x} - \omega\frac{\partial}{\partial t} \tag{101}$$

and

$$\frac{\partial\boldsymbol{v}}{\partial t} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}\frac{\partial^2\boldsymbol{v}}{\partial x^2}. \tag{102}$$

The invariant solutions under the symmetries (101) have the form

$$v_1 = \mathrm{e}^{-\alpha_1 x}\varphi_1(\omega_1 - a_1 x) = \mathrm{e}^{-\alpha_1 x}\varphi_1(\xi_1) \tag{103}$$

$$v_2 = \mathrm{e}^{-\alpha_2 x}\varphi_2(\omega_2 - a_2 x) = \mathrm{e}^{-\alpha_2 x}\varphi_2(\xi_2) \tag{104}$$

and so substitution of (103), (104) in (102) means that

$$\frac{\omega}{\lambda_1}\varphi_1' = \alpha_1^2[\varphi_1 + 2\varphi_1' + \varphi_1''] \qquad \frac{\omega}{\lambda_2}\varphi_2' = \alpha_2^2[\varphi_2 + 2\varphi_2' + \varphi_2''] \tag{105}$$

with solutions

$$\varphi_1 = \mathrm{Re}\,(C_1\mathrm{e}^{\gamma_1\xi_1} + C_2\mathrm{e}^{\gamma_2\xi_2}) \qquad \varphi_2 = \mathrm{Re}\,(C_3\mathrm{e}^{\tau_1\xi_2} + C_4\mathrm{e}^{\tau_2\xi_2}) \tag{106}$$

$$\begin{aligned}\gamma_{1,2} &= -1 + \frac{\omega}{2\lambda_1\alpha_1^2} \pm \sqrt{\left(1 - \frac{\omega}{2\lambda_1\alpha_1^2}\right)^2 - 1}\\ \tau_{1,2} &= -1 + \frac{\omega}{2\lambda_2\alpha_2^2} \pm \sqrt{\left(1 - \frac{\omega}{2\lambda_2\alpha_2^2}\right)^2 - 1}.\end{aligned} \tag{107}$$

If $\alpha_i = \sqrt{\omega/2\lambda_i},\ i = 1, 2$ then $\gamma_{1,2} = \pm\mathrm{i}$, $\tau_{1,2} = \pm\mathrm{i}$ and so

$$\varphi_1 = a_1\cos(\xi_1) + b_1\sin(\xi_1) \qquad \varphi_2 = a_2\cos(\xi_2) + b_2\sin(\xi_2). \tag{108}$$

With $b_i = 0$ the solution is

$$\boldsymbol{y}(x,t) = a_j\mathrm{e}^{-\alpha_j x}\cos(\omega t - \alpha_j x)\boldsymbol{n}_j. \tag{109}$$

This solution for coupled diffusion was obtained by Shepherd and Wiltshire [16], where $\boldsymbol{y}(x,t)$ is the variation about the mean value and where the repeated index implies summation from 1 to 2.

In addition, if $\Lambda^{[0]} = \Lambda_J[0]$ then

$$\begin{aligned}\boldsymbol{n}_1 &= \begin{pmatrix} 0.9\dot{9} \\ 2.44\times10^{-5}\end{pmatrix} \qquad \lambda_1 = 2.23\times10^{-3}\\ \boldsymbol{n}_2 &= \begin{pmatrix} -0.89 \\ 0.46\end{pmatrix} \qquad \lambda_2 = 5.06\times10^{-5}.\end{aligned} \tag{110}$$

6.2. *Quasi-constant $\Lambda(\boldsymbol{y})$*

The question arises as to how this solution may be modified for instances when Λ is no longer constant and yet harmonic boundary conditions are still required. From the point of view of similarity reduction, consideration is given to analytic forms of $\Lambda(\boldsymbol{y})$, including constant $\Lambda^{[0]}$, for which travelling wave solutions exist, which also satisfy specified harmonic boundary conditions. However, the simple use of the similarity variable $\xi = \mu_1 t + \mu_2 x$ will not automatically give rise to solutions proportional to $\cos(\omega t)$ on the boundary. Indeed, even in cases where μ_1 is complex, solutions with harmonic boundary conditions cannot be obtained unless two similarity variables $\xi_1 = \mathrm{i}\omega t + \mu_2 x$ and $\xi = -\mathrm{i}\omega t + \mu_2 x$ are employed. The approach adopted here will be to write our travelling wave solution in the form $\boldsymbol{y}(x,t) = f(x)\boldsymbol{\Phi}(\xi)$, $\xi = \omega t - \alpha x$. In fact, it is easy to see on multiplication of (2) by $\mathrm{e}^{\omega t}$ that the following ordinary differential equation:

$$\omega\mathrm{e}^{\xi}\frac{\mathrm{d}\boldsymbol{\Phi}}{\mathrm{d}\xi} = \alpha\frac{\mathrm{d}}{\mathrm{d}\xi}\left[\Lambda\alpha\frac{\mathrm{d}}{\mathrm{d}\xi}(\mathrm{e}^{\xi}\boldsymbol{\Phi})\right] \tag{111}$$

is obtained where now

$$\boldsymbol{y}(x,t) = \mathrm{e}^{-\alpha x}\boldsymbol{\Phi}(\xi). \tag{112}$$

As suggested by the above symmetry classification analysis it will be supposed that $\Lambda(\boldsymbol{y})$ has a form given by equation (68). In addition it will also be assumed that $k = 0$ so that the case of constant Λ is included as a limiting case of this analysis. In addition, using equation (84) it follows that

$$\Lambda(\boldsymbol{y}) = \Lambda\left(\frac{y_2}{y_1}\right) \equiv \Lambda\big(f(\xi)\big). \tag{113}$$

However, equation (111) must be modified to include the fact that the proposed solutions $\boldsymbol{y}$, modifications of (109), have two components $\boldsymbol{n}_i,\ i = 1, 2$, and are also functions of the two similarity variables ξ_1 and ξ_2. In particular, equations (112), (113) will be generalized so that

$$\begin{aligned} \xi_i &= \omega t - \alpha_i x \\ \boldsymbol{y}(x,t) &= \mathrm{e}^{-\alpha i x}\boldsymbol{\Phi}_i(\xi_1,\xi_2) \\ \Lambda\left(\frac{y_2}{y_1}\right) &= \Lambda\big(f(\xi_1), g(\xi_2)\big). \end{aligned} \tag{114}$$

In this way equation (2) may be written as

$$\omega_j \mathrm{e}^{\xi_i}\frac{\partial\boldsymbol{\Phi}_i}{\partial\xi_j} = \alpha_k\frac{\partial}{\partial\xi_k}\left[\Lambda\alpha_i\frac{\partial}{\partial\xi_i}(\mathrm{e}^{\xi_j}\boldsymbol{\Phi}_j)\right] \tag{115}$$

where again repeated indices imply summation from 1 to 2, and further

$$\omega_j = \omega \qquad j = 1, 2. \tag{116}$$

Of course, equation (115) could be analysed in detail using the method presented in section 5. However, the approach here will be to adopt a perturbation method to account for small variations in $\Lambda(\boldsymbol{y})$ as described by Jackson [6] and Rose [14]. Thus it will be assumed that

$$\begin{aligned} \Lambda\big(f(\xi_1), g(\xi_2)\big) &= \Lambda^{[0]} + \varepsilon\Lambda^{[1]}(\xi_1,\xi_2) + \mathrm{O}(\varepsilon^2) \\ \boldsymbol{\Phi}_i(\xi_1,\xi_2) &= \boldsymbol{\Phi}_i^{[0]}(\xi_1,\xi_2) + \varepsilon\boldsymbol{\Phi}_i^{[1]}(\xi_1,\xi_2) + \mathrm{O}(\varepsilon^2) \\ \boldsymbol{\Phi}_i^{[0]}(\xi_1,\xi_2) &= a_i\cos\xi_i\boldsymbol{n}_i. \end{aligned} \tag{117}$$

In this way, it may be shown that the equation arising from coefficients of ε^0 gives rise exactly to the solution (109), whilst the equation for component $\boldsymbol{\Phi}_j^{[1]}$ arising from ε^1 is

$$\Lambda^{[0]}\alpha_i\alpha_k\frac{\partial^2}{\partial\xi_i\partial\xi_k}(\mathrm{e}^{\xi_j}\boldsymbol{\Phi}_j^{[1]}) = \omega_k\mathrm{e}^{\xi_j}\frac{\partial\boldsymbol{\Phi}_j^{[1]}}{\partial\xi_k} - \alpha_k\frac{\partial}{\partial\xi_k}\left[\Lambda^{[1]}\alpha_i\frac{\partial}{\partial\xi_i}(\mathrm{e}^{\xi_j}\boldsymbol{\Phi}_j^{[0]})\right]. \tag{118}$$

This equation may be simplified in cases of (113) where

$$\boldsymbol{\Phi}_i^{[1]} = \phi_i^{[1]}\boldsymbol{n}_i \qquad \Lambda^{[1]} = \Lambda^{[0]}\sigma(\xi_1,\xi_2) \tag{119}$$

to give the linear differential equation for component $\phi_j^{[1]},\ j = 1, 2$:

$$\boldsymbol{n}_j\lambda_j\alpha_i\alpha_k\frac{\partial^2}{\partial\xi_i\partial\xi_k}(\mathrm{e}^{\xi_j}\phi_j^{[1]}) = \omega_k\mathrm{e}^{\xi_i}\frac{\partial\phi_j^{[1]}}{\partial\xi_k}\boldsymbol{n}_j - \alpha_k\frac{\partial}{\partial\xi_k}\big[\sigma(\xi_1,\xi_2)\lambda_j\mathrm{e}^{\xi_j}(\cos\xi_j - \sin\xi_j)\big]\boldsymbol{n}_j. \tag{120}$$

This equation may be solved for specific $\sigma(\xi_1, \xi_2)$. In particular, when

$$\sigma(\xi_1, \xi_2) = \sigma_0 \exp[\rho_1\xi_1 + \rho_2\xi_2] \tag{121}$$

then

$$\phi_j^{[1]}(\xi_1, \xi_2) = X_j(\xi_j) \exp[\rho_1\xi_1 + \rho_2\xi_2] \tag{122}$$

where

$$X_j(\xi_j) = \mathrm{e}^{-\eta_j\xi_j}\left\{A_j \exp\big[\sqrt{\beta_j}\xi_1\big] + B_j \exp\big[-\sqrt{\beta_j}\xi_1\big]\right\}$$
$$+\frac{a_j\sigma_0}{s_j^2 + 4\eta_j^2}\left\{\big[s_j(\eta_j + 2) - 2\eta_j^2\big]\sin\xi_1 - \eta_j\big[s_j + 2\eta_j + 4\big]\cos\xi_1\right\} \tag{123}$$

and

$$\begin{aligned} \eta_j &= \frac{\alpha_1\rho_1 + \alpha_2\rho_2}{\alpha_j} > 0 \qquad \beta_1 = 2\rho_2\left(\frac{\alpha_1 - \alpha_2}{\alpha_1}\right) - 1 \\ s_j &= \eta_j^2 + \beta_j - 1 \qquad \beta_2 = 2\rho_1\left(\frac{\alpha_2 - \alpha_1}{\alpha_2}\right) - 1. \end{aligned} \tag{124}$$

In summary, a final solution to first order in ε when

$$\Lambda\big(f(\xi_1), g(\xi_2)\big) = \Lambda^{[0]}\big(1 + \varepsilon\sigma_0 \exp[\rho_1\xi_1 + \rho_2\xi_2]\big) \tag{125}$$

is

$$\boldsymbol{y}(x, t) = \mathrm{e}^{-\alpha_j x}\left(a_j \cos\xi_j + \varepsilon X_j(\xi_j) \exp[\rho_1\xi_1 + \rho_2\xi_2]\right)\boldsymbol{n}_j \tag{126}$$

where $X_j(\xi_j)$is given by (123). If now the constants A_j, B_j are chosen so that $\phi_j^{[1]}(0, 0) = 0$ in (123), then equation (126) will satisfy the harmonic boundary conditions.

In this solution the matrix of diffusion coefficients is a nonlinear function of the rations of the components of $\boldsymbol{y}$ in such a way that the surfaxes $\xi_i =$ constant give rise to a constant value of Λ. Clearly, the perturbation approach may be extended to higher orders in ε and also a more detailed analysis of potential analytic forms for Λ is required to include a broader range of possibilities than the exponential example of (121).

Finally, numerical calculations of (125) and (121) are straightforward. The eigenvalues and eigenvectors of $\Lambda^{[0]}$ may be calculated from (96) so that α_i will be known for annual or diurnal variation. In addition, the amplitudes of temperature and moisture variation may be taken from the experiments of Jackson [6] or Rose [14], so for example $[\,a_1 \quad a_2\,]^{\mathrm{T}} \approx [\,10 \quad 0.1\,]^{\mathrm{T}}$, and further $\sigma_0 \approx 10^{-2}$ appears to be valid. It follows that the spatial distributions and time evolutions of $\boldsymbol{y}(x, t)$ may be plotted for particular cases of ρ_1 and ρ_2.

Practical situations pose further interesting mathematical problems. The combination on the boundary of diurnal variations in temperature and impulsive inputs for moisture is of particular note. More generally, only Dirichlet boundary conditions have been considered here, although in practical cases both Neumann and Robin situations are also particularly important.

Appendix. Generation of the determining equations

The specification of the prolongation operator (4) can be completed using

$$\boldsymbol{\pi}_x = \big[\boldsymbol{\pi}' + (\boldsymbol{y}'\cdot\nabla)\boldsymbol{\pi}\big] - \big[\xi' + (\boldsymbol{y}'\cdot\nabla)\xi\big]\boldsymbol{y}' - \big[\eta' + (\boldsymbol{y}'\cdot\nabla)\eta\big]\dot{\boldsymbol{y}} \tag{A1}$$

$$\boldsymbol{\pi}_t = \big[\dot{\boldsymbol{\pi}} + (\dot{\boldsymbol{y}}\boldsymbol{\cdot}\nabla)\boldsymbol{\pi}\big] - \big[\dot{\xi} + (\dot{\boldsymbol{y}}\boldsymbol{\cdot}\nabla)\xi\big]\boldsymbol{y}' - \big[\dot{\eta} + (\dot{\boldsymbol{y}}\boldsymbol{\cdot}\nabla)\eta\big]\dot{\boldsymbol{y}} \tag{A2}$$

$$\begin{aligned}
\boldsymbol{\pi}_{xx} = {}& \big[\boldsymbol{\pi}'' + (\boldsymbol{y}'\boldsymbol{\cdot}\nabla)\boldsymbol{\pi}' + (\boldsymbol{y}''\boldsymbol{\cdot}\nabla)\boldsymbol{\pi}\big] - \big[\xi'' + (\boldsymbol{y}'\boldsymbol{\cdot}\nabla)\xi' + (\boldsymbol{y}''\boldsymbol{\cdot}\nabla)\xi\big]\boldsymbol{y}' \\
& - \big[\eta'' + (\boldsymbol{y}'\boldsymbol{\cdot}\nabla)\eta' + (\boldsymbol{y}''\boldsymbol{\cdot}\nabla)\eta\big]\dot{\boldsymbol{y}} + (\boldsymbol{y}'\boldsymbol{\cdot}\nabla)\boldsymbol{\pi}' + (\boldsymbol{y}'\boldsymbol{\cdot}\nabla)^2\boldsymbol{\pi} \\
& - \big[(\boldsymbol{y}'\boldsymbol{\cdot}\nabla)\xi' + (\boldsymbol{y}'\boldsymbol{\cdot}\nabla)^2\xi\big]\boldsymbol{y}' - \big[(\boldsymbol{y}'\boldsymbol{\cdot}\nabla)\eta' + (\boldsymbol{y}'\boldsymbol{\cdot}\nabla)^2\eta\big]\dot{\boldsymbol{y}} \\
& - 2\big[\xi' + (\boldsymbol{y}'\boldsymbol{\cdot}\nabla)\xi\big]\boldsymbol{y}'' - 2\big[\eta' + (\boldsymbol{y}'\boldsymbol{\cdot}\nabla)\eta\big]\dot{\boldsymbol{y}}'.
\end{aligned} \tag{A3}$$

Therefore the condition for invariance $\mathcal{X}_E\, \boldsymbol{p}|_{(2)} = \boldsymbol{0}$ becomes

$$\boldsymbol{\pi}_t - \Lambda\boldsymbol{\pi}_{xx} - \big[(\boldsymbol{\pi}\boldsymbol{\cdot}\nabla)\Lambda\big]\boldsymbol{y}'' - \big[(\boldsymbol{y}'\boldsymbol{\cdot}\nabla)\Lambda\big]\boldsymbol{\pi}_x - \big[(\boldsymbol{\pi}_x\boldsymbol{\cdot}\nabla)\Lambda\big]\boldsymbol{y}' - \big[(\boldsymbol{\pi}\boldsymbol{\cdot}\nabla)(\boldsymbol{y}'\boldsymbol{\cdot}\nabla)\Lambda\big]\boldsymbol{y}'\big|_{(2)} = 0. \tag{A4}$$

In addition, $\dot{\boldsymbol{y}}$ may be eliminated using (2) in the form

$$\dot{\boldsymbol{y}} = \Lambda\boldsymbol{y}'' + \big[(\boldsymbol{y}'\boldsymbol{\cdot}\nabla)\Lambda\big]\boldsymbol{y}'. \tag{A5}$$

In order to find symmetry groups, the coefficients of (A4) must be equated to give the following conditions:

Constant: $\dot{\boldsymbol{\pi}} = \Lambda\boldsymbol{\pi}''$

$\boldsymbol{y}'$: $-\dot{\xi}\boldsymbol{y}' = 2\Lambda(\boldsymbol{y}'\boldsymbol{\cdot}\nabla)\boldsymbol{\pi}' - \xi''\Lambda\boldsymbol{y}' + \big[(\boldsymbol{y}'\boldsymbol{\cdot}\nabla)\Lambda\big]\boldsymbol{\pi}' + \big[(\boldsymbol{\pi}'\boldsymbol{\cdot}\nabla)\Lambda\big]\boldsymbol{y}'$

$\boldsymbol{y}''$: $(\Lambda\boldsymbol{y}''\boldsymbol{\cdot}\nabla)\boldsymbol{\pi} - \dot{\eta}\Lambda\boldsymbol{y}'' = \Lambda\big[(\boldsymbol{y}''\boldsymbol{\cdot}\nabla)\boldsymbol{\pi}\big] - \Lambda\eta''\Lambda\boldsymbol{y}'' - 2\xi'\Lambda\boldsymbol{y}'' + \big[(\boldsymbol{\pi}\boldsymbol{\cdot}\nabla)\Lambda\big]\boldsymbol{y}''$

$\boldsymbol{y}', \boldsymbol{y}'$:

$$\begin{aligned}
&\big(\big[(\boldsymbol{y}'\boldsymbol{\cdot}\nabla)\Lambda\big]\boldsymbol{y}'\boldsymbol{\cdot}\nabla\big)\,\boldsymbol{\pi} - \dot{\eta}\big[(\boldsymbol{y}'\boldsymbol{\cdot}\nabla)\Lambda\big]\boldsymbol{y}' \\
&\quad = -\Lambda(\boldsymbol{y}'\boldsymbol{\cdot}\nabla)\xi'\boldsymbol{y}' - \Lambda\eta''\big[(\boldsymbol{y}'\boldsymbol{\cdot}\nabla)\Lambda\big]\boldsymbol{y}' + \Lambda(\boldsymbol{y}'\boldsymbol{\cdot}\nabla)^2\boldsymbol{\pi} \\
&\qquad - \Lambda(\boldsymbol{y}'\boldsymbol{\cdot}\nabla)\xi'\boldsymbol{y}' + \big[(\boldsymbol{y}'\boldsymbol{\cdot}\nabla)\Lambda\big]\big[(\boldsymbol{y}'\boldsymbol{\cdot}\nabla)\boldsymbol{\pi}\big] - \big[(\boldsymbol{y}'\boldsymbol{\cdot}\nabla)\Lambda\big]\xi'\boldsymbol{y}' \\
&\qquad + \big[\big(\big\{\big[(\boldsymbol{y}'\boldsymbol{\cdot}\nabla)\boldsymbol{\pi}\big] - \xi'\boldsymbol{y}'\big\}\boldsymbol{\cdot}\nabla\big)\,\Lambda\big]\,\boldsymbol{y}' + \big[(\boldsymbol{\pi}\boldsymbol{\cdot}\nabla)(\boldsymbol{y}'\boldsymbol{\cdot}\nabla)\Lambda\big]\boldsymbol{y}'
\end{aligned}$$

$\boldsymbol{y}', \boldsymbol{y}''$:

$$\begin{aligned}
-\big[(\Lambda\boldsymbol{y}''\boldsymbol{\cdot}\nabla)\xi\big]\boldsymbol{y}' = {}& -\Lambda(\boldsymbol{y}''\boldsymbol{\cdot}\nabla)\xi\boldsymbol{y}' - \Lambda(\boldsymbol{y}'\boldsymbol{\cdot}\nabla)\eta'\Lambda\boldsymbol{y}'' - \Lambda(\boldsymbol{y}'\boldsymbol{\cdot}\nabla)\eta'\Lambda\boldsymbol{y}'' \\
& - 2\Lambda\big[(\boldsymbol{y}'\boldsymbol{\cdot}\nabla)\xi\big]\boldsymbol{y}'' - \big[(\boldsymbol{y}'\boldsymbol{\cdot}\nabla)\Lambda\big]\eta'\Lambda\boldsymbol{y}'' - \big[(\eta'\Lambda\boldsymbol{y}''\boldsymbol{\cdot}\nabla)\Lambda\big]\boldsymbol{y}'
\end{aligned}$$

$\boldsymbol{y}'', \boldsymbol{y}''$: $-\big[(\Lambda\boldsymbol{y}''\boldsymbol{\cdot}\nabla)\eta\big]\Lambda\boldsymbol{y}'' = -\Lambda\big[(\boldsymbol{y}''\boldsymbol{\cdot}\nabla)\eta\big]\Lambda\boldsymbol{y}''$

$\boldsymbol{y}', \boldsymbol{y}', \boldsymbol{y}'$:

$$\begin{aligned}
-\big[\big(\big[(\boldsymbol{y}'\boldsymbol{\cdot}\nabla)\Lambda\big]\boldsymbol{y}'\boldsymbol{\cdot}\nabla\big)\,\xi\big]\,\boldsymbol{y}' = {}& -\Lambda\big[(\boldsymbol{y}'\boldsymbol{\cdot}\nabla)\eta'\big]\,\big(\big[(\boldsymbol{y}'\boldsymbol{\cdot}\nabla)\Lambda\big]\boldsymbol{y}'\big) - \Lambda(\boldsymbol{y}'\boldsymbol{\cdot}\nabla)^2\xi\boldsymbol{y}' \\
& - \Lambda(\boldsymbol{y}'\boldsymbol{\cdot}\nabla)\eta'\big[(\boldsymbol{y}'\boldsymbol{\cdot}\nabla)\Lambda\big]\boldsymbol{y}' - \big[(\boldsymbol{y}'\boldsymbol{\cdot}\nabla)\Lambda\big](\boldsymbol{y}'\boldsymbol{\cdot}\nabla)\xi\boldsymbol{y}' \\
& - \big[\big(\big\{(\boldsymbol{y}'\boldsymbol{\cdot}\nabla)\xi\boldsymbol{y}' - \big[\eta' + (\boldsymbol{y}'\boldsymbol{\cdot}\nabla)\eta\big]\big[(\boldsymbol{y}'\boldsymbol{\cdot}\nabla)\Lambda\big]\boldsymbol{y}'\big\}\boldsymbol{\cdot}\nabla\big)\,\Lambda\big]\,\boldsymbol{y}'
\end{aligned}$$

$\boldsymbol{y}', \boldsymbol{y}', \boldsymbol{y}''$:

$$\begin{aligned}
&-\big[(\Lambda\boldsymbol{y}''\boldsymbol{\cdot}\nabla)\eta\big]\big[(\boldsymbol{y}'\boldsymbol{\cdot}\nabla)\Lambda\big]\boldsymbol{y}' - \big[\big(\big[(\boldsymbol{y}'\boldsymbol{\cdot}\nabla)\Lambda\big]\boldsymbol{y}'\boldsymbol{\cdot}\nabla\big)\,\eta\big]\,\Lambda\boldsymbol{y}'' \\
&\quad = -\Lambda\big[(\boldsymbol{y}''\boldsymbol{\cdot}\nabla)\eta\big]\,\big(\big[(\boldsymbol{y}'\boldsymbol{\cdot}\nabla)\Lambda\big]\boldsymbol{y}'\big) - \Lambda\big[(\boldsymbol{y}'\boldsymbol{\cdot}\nabla)^2\eta\big]\Lambda\boldsymbol{y}'' \\
&\qquad - \big[(\boldsymbol{y}'\boldsymbol{\cdot}\nabla)\Lambda\big]\big[(\boldsymbol{y}'\boldsymbol{\cdot}\nabla)\eta\big]\Lambda\boldsymbol{y}'' + \big[\big((\boldsymbol{y}'\boldsymbol{\cdot}\nabla)\eta\Lambda\boldsymbol{y}''\boldsymbol{\cdot}\nabla\big)\Lambda\big]\,\boldsymbol{y}'
\end{aligned}$$

$\boldsymbol{y}', \boldsymbol{y}', \boldsymbol{y}', \boldsymbol{y}'$:

$$\begin{aligned}
&-\big[\big(\big[(\boldsymbol{y}'\boldsymbol{\cdot}\nabla)\Lambda\big]\boldsymbol{y}'\boldsymbol{\cdot}\nabla\big)\,\eta\big]\,\big[(\boldsymbol{y}'\boldsymbol{\cdot}\nabla)\Lambda\big]\boldsymbol{y}' \\
&\quad = -\Lambda\big[(\boldsymbol{y}'\boldsymbol{\cdot}\nabla)^2\eta\big]\big[(\boldsymbol{y}'\boldsymbol{\cdot}\nabla)\Lambda\big]\boldsymbol{y}'
\end{aligned}$$

$$-\big[(\boldsymbol{y}'\cdot\nabla)\Lambda\big]\big[(\boldsymbol{y}'\cdot\nabla)\eta\big]\big[(\boldsymbol{y}'\cdot\nabla)\Lambda\big]\boldsymbol{y}'$$

$\dot{\boldsymbol{y}}'$: $\quad 0 = -2\Lambda\eta'\dot{\boldsymbol{y}}'$

$\boldsymbol{y}', \dot{\boldsymbol{y}}'$: $\quad 0 = -2\Lambda(\boldsymbol{y}'\cdot\nabla)\eta\dot{\boldsymbol{y}}'$.

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